

# Who is the Lord of the Rings: Majorana, Dirac or Lifshitz? The Spin-Orbit-Zeeman Saga in Ultra-cold Fermions.

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We discuss the emergence of rings of zero-energy excitations in momentum space for superfluid phases of ultra-cold fermions when spin-orbit, Zeeman fields and interactions are varied. We show that phases containing rings of nodes possess non-trivial topological invariants, and that phase transitions between distinct topological phases belong to the Lifshitz class. Upon crossing phase boundaries, existing massless Dirac fermions in the gapless phase annihilate to produce bulk zero-mode Majorana fermions at phase boundaries and then become massive Dirac fermions in the gapped phase. We characterize these tunable topological phase transitions via several spectroscopic properties, including excitation spectrum, spectral function and momentum distribution. Since the emergence or disappearance of rings leads to topological transitions in momentum space, we conclude that Lifshitz is the lord of the rings.

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Ultra-cold atoms have now become standard laboratories to test for existing or new theoretical ideas in atomic, condensed matter, nuclear and astrophysics. The major appeal found in these table-top experiments is the ability to tune interactions, populations, species of atoms and dimensionality - which constitute the standard toolbox for investigations of interacting bosonic or fermionic systems. Very recently a new tool has been added to the toolbox: the ability to tune simultaneously spin-orbit and Zeeman fields in a system of ultra-cold bosonic atoms [1]. The same tool can also be used to study ultra-cold fermionic atoms [1–3] and to simulate different condensed matter systems such as topological insulators [4], non-centrosymmetric superconductors [5] and non-equilibrium systems [6], where spin-orbit coupling of the Rashba-type [7] is encountered.

This direct connection to condensed matter physics inspired a new direction in ultra-cold fermionic atoms where spin-orbit coupling of the Rashba-type has been very recently investigated [8–13]. However, spin-orbit fields currently realized in experiments involving ultra-cold atoms correspond to an equal superposition of Rashba [7]  $\mathbf{h}_R(\mathbf{k}) = v_R(-k_y\hat{\mathbf{x}} + k_x\hat{\mathbf{y}})$  and Dresselhaus [14]  $\mathbf{h}_D(\mathbf{k}) = v_D(k_y\hat{\mathbf{x}} + k_x\hat{\mathbf{y}})$  fields, leading to equal-Rashba-Dresselhaus (ERD) form [1, 13]  $\mathbf{h}_{ERD}(\mathbf{k}) = v k_x \hat{\mathbf{y}}$ , where  $v_R = v_D = v/2$ . Other forms of spin-orbit fields require additional lasers and create further experimental difficulties [15], such that the current Zeeman-spin-orbit Hamiltonian created in the laboratory is

$$\mathbf{H}_{ZSO}(\mathbf{k}) = -h_z\sigma_z - h_y\sigma_y - h_{ERD}(\mathbf{k})\sigma_y \quad (1)$$

for an atom with center-of-mass momentum  $\mathbf{k}$  and spin basis  $|\uparrow\rangle, |\downarrow\rangle$ . The fields  $h_z = -\Omega_R/2$ ,  $h_y = -\delta/2$ , and  $h_{ERD}(\mathbf{k}) = v k_x$  can be controlled independently and can be used to explore phase diagrams as achieved in  $^{87}\text{Rb}$  experiments [1]. Here,  $\Omega_R$  is the Raman coupling and  $\delta$  is the detuning.

*Hamiltonian:* To investigate artificial spin-orbit and Zeeman fields in ultra-cold Fermi superfluids, we start from the Hamiltonian density  $\mathcal{H}(\mathbf{r}) = \mathcal{H}_0(\mathbf{r}) + \mathcal{H}_I(\mathbf{r})$ , with  $\hbar = 1$ . The single-particle contribution is

$$\mathcal{H}_0(\mathbf{r}) = \sum_{s,s'} \psi_s^\dagger(\mathbf{r}) \left[ K(\hat{\mathbf{k}}) \mathbf{1} + \mathbf{H}_{ZSO}(\hat{\mathbf{k}}) \right]_{ss'} \psi_{s'}(\mathbf{r}), \quad (2)$$

where  $K(\hat{\mathbf{k}}) = \hat{\mathbf{k}}^2/(2m) - \mu$  is the kinetic energy relative to the chemical potential  $\mu$ ,  $[\mathbf{H}_{ZSO}(\hat{\mathbf{k}})]_{ss'}$  are the matrix elements of the Zeeman-spin-orbit matrix Hamiltonian defined in Eq. (1),  $\hat{\mathbf{k}} = -i\nabla$  is the momentum operator, and  $\psi_s^\dagger(\mathbf{r})$  creates fermions with spin  $s$  at position  $\mathbf{r}$ . The interaction term

$$\mathcal{H}_I(\mathbf{r}) = -g\psi_\uparrow^\dagger(\mathbf{r})\psi_\downarrow^\dagger(\mathbf{r})\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r}), \quad (3)$$

is local and  $g$  represents a contact interaction strength. We define the total number of fermions as  $N = N_\uparrow + N_\downarrow$ , and the induced population imbalance as  $P_{\text{ind}} = (N_\uparrow - N_\downarrow)/N$ . We choose our scales through the Fermi momentum  $k_F$  defined from the total density of fermions  $n = n_\uparrow + n_\downarrow = k_F^3/(3\pi^2)$ . This choice leads to the Fermi energy  $\epsilon_F = k_F^2/2m$  and to the Fermi velocity  $v_F = k_F/m$ , as energy and velocity scales respectively.

We focus on the zero-detuning case  $\delta = 0$  ( $h_y = 0$ ), use the basis  $\psi_\uparrow^\dagger(\mathbf{k})|0\rangle \equiv |\mathbf{k}\uparrow\rangle$ ,  $\psi_\downarrow^\dagger(\mathbf{k})|0\rangle \equiv |\mathbf{k}\downarrow\rangle$ , where  $|0\rangle$  is the vacuum state, and write  $\mathcal{H}_0(\mathbf{r})$  as the matrix

$$\mathbf{H}_0(\mathbf{k}) = K(\mathbf{k})\mathbf{1} - h_z\sigma_z - h_{ERD}(\mathbf{k})\sigma_y. \quad (4)$$

The interaction Hamiltonian  $\mathcal{H}_I(\mathbf{r})$  can also be converted into momentum space as  $\mathcal{H}_I(\mathbf{q}) = -gb^\dagger(\mathbf{q})b(\mathbf{q})$ , where the pair creation operator with center of mass momentum  $\mathbf{q}$  is  $b^\dagger(\mathbf{q}) = \sum_{\mathbf{k}} \psi_\uparrow^\dagger(\mathbf{k} + \mathbf{q}/2)\psi_\downarrow^\dagger(-\mathbf{k} + \mathbf{q}/2)$ , and  $g$  can be expressed in terms of the scattering length through  $V/g = -Vm/(4\pi a_s) + \sum_{\mathbf{k}} 1/(2\epsilon_{\mathbf{k}})$ .

*Helicity Basis:* The matrix  $\mathbf{H}_0(\mathbf{k})$  can be diagonalized in the helicity basis  $\Phi_{\uparrow\uparrow}^\dagger(\mathbf{k})|0\rangle \equiv |\mathbf{k}\uparrow\uparrow\rangle$ ,  $\Phi_{\uparrow\downarrow}^\dagger(\mathbf{k})|0\rangle \equiv |\mathbf{k}\uparrow\downarrow\rangle$ ,

via a momentum-dependent SU(2) rotation. The helicity spins  $\uparrow\downarrow$  and  $\downarrow\uparrow$  are aligned or antialigned with respect to the effective magnetic field  $\mathbf{h}_{\text{eff}}(\mathbf{k}) = h_z \hat{\mathbf{z}} + h_{\text{ERD}}(\mathbf{k}) \hat{\mathbf{y}}$ . The eigenvalues of the Hamiltonian matrix  $\mathbf{H}_0(\mathbf{k})$  are  $\xi_{\uparrow\uparrow}(\mathbf{k}) = K(\mathbf{k}) - |\mathbf{h}_{\text{eff}}(\mathbf{k})|$  and  $\xi_{\downarrow\downarrow}(\mathbf{k}) = K(\mathbf{k}) + |\mathbf{h}_{\text{eff}}(\mathbf{k})|$ , where  $|\mathbf{h}_{\text{eff}}(\mathbf{k})| = \sqrt{h_z^2 + h_{\text{ERD}}^2(\mathbf{k})}$  is the magnitude of the effective magnetic field. The interaction Hamiltonian  $\mathcal{H}_I(\mathbf{q})$  can be written in the helicity basis as  $\tilde{\mathcal{H}}_I(\mathbf{q}) = -g \sum_{\alpha\beta\gamma\delta} B_{\alpha\beta}^\dagger(\mathbf{q}) B_{\gamma\delta}(\mathbf{q})$ , where pairing is now described by the operator

$$B_{\alpha\beta}^\dagger(\mathbf{q}) = \sum_{\mathbf{k}} \Lambda_{\alpha\beta}(\mathbf{k}_1, \mathbf{k}_2) \Phi_\alpha^\dagger(\mathbf{k}_1) \Phi_\beta^\dagger(\mathbf{k}_2) \quad (5)$$

and its Hermitian conjugate, with momentum indices  $\mathbf{k}_1 = \mathbf{k} + \mathbf{q}/2$  and  $\mathbf{k}_2 = -\mathbf{k} + \mathbf{q}/2$ . The matrix  $\Lambda_{\alpha\beta}(\mathbf{k} + \mathbf{q}/2, -\mathbf{k} + \mathbf{q}/2)$  is directly related to the matrix elements of the momentum dependent SU(2) rotation into the helicity basis, and reveals that the center of mass momentum  $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{q}$  and the relative momentum  $\mathbf{k}_1 - \mathbf{k}_2 = 2\mathbf{k}$  are coupled and no longer independent.

*Tensor Order Parameter:* From Eq. (5) it is clear that pairing between fermions of momenta  $\mathbf{k}_1$  and  $\mathbf{k}_2$  can occur within the same helicity band (intra-helicity pairing) or between two different helicity bands (inter-helicity pairing). For pairing at  $\mathbf{q} = 0$ , the order parameter for superfluidity is the tensor  $\Delta_{\alpha\beta}(\mathbf{k}) = \Delta_0 \Lambda_{\alpha\beta}(\mathbf{k}, -\mathbf{k})$ , where  $\Delta_0 = -g \sum_{\gamma\delta} \langle B_{\gamma\delta}(\mathbf{0}) \rangle$ , leading to components:  $\Delta_{\uparrow\uparrow}(\mathbf{k}) = i\Delta_T(\mathbf{k}) \text{sgn}[k_x]$  for helicity projection  $\lambda = +1$ ;  $\Delta_{\uparrow\downarrow}(\mathbf{k}) = -\Delta_S(\mathbf{k})$ , and  $\Delta_{\downarrow\uparrow}(\mathbf{k}) = \Delta_S(\mathbf{k})$ , for helicity projection  $\lambda = 0$ ; and  $\Delta_{\downarrow\downarrow}(\mathbf{k}) = -i\Delta_T(\mathbf{k}) \text{sgn}[k_x]$ , for helicity projection  $\lambda = -1$ . The amplitudes  $\Delta_T(\mathbf{k}) = \Delta_0 |h_{\text{ERD}}(\mathbf{k})| / |\mathbf{h}_{\text{eff}}(\mathbf{k})|$  and  $\Delta_S(\mathbf{k}) = \Delta_0 h_z / |\mathbf{h}_{\text{eff}}(\mathbf{k})|$  reflect the triplet and singlet components of the order parameter in the helicity basis. The Bloch-sphere relation  $|\Delta_T(\mathbf{k})|^2 + |\Delta_S(\mathbf{k})|^2 = |\Delta_0|^2$ , shows that the singlet and triplet channels in the helicity basis are not independent.

*Higher angular momentum pairing:* In the triplet sector  $\Delta_{\uparrow\uparrow}(\mathbf{k})$  and  $\Delta_{\downarrow\downarrow}(\mathbf{k})$  contain not only  $p$ -wave, but also  $f$ -wave and higher odd partial waves, as seen from a multipole expansion of  $|\mathbf{h}_{\text{eff}}(\mathbf{k})|^{-1} = [h_z^2 + h_{\text{ERD}}^2(\mathbf{k})]^{-1/2}$  for finite  $h_z$ . Similarly in the singlet sector  $\Delta_{\uparrow\downarrow}(\mathbf{k})$  and  $\Delta_{\downarrow\uparrow}(\mathbf{k})$  contain  $s$ -wave,  $d$ -wave and higher even partial waves, as long as the Zeeman field  $h_z$  is non-zero. Higher angular momentum pairing occurs because the local (zero-ranged) interaction in the  $(\uparrow, \downarrow)$  spin basis is transformed into a finite-ranged anisotropic interaction in the helicity basis  $(\uparrow\downarrow, \downarrow\uparrow)$ .

*Excitation Spectrum:* The effective Hamiltonian in the helicity basis takes the matrix form

$$\tilde{\mathbf{H}}_{\text{sp}}(\mathbf{k}) = \begin{pmatrix} \xi_{\uparrow\uparrow}(\mathbf{k}) & 0 & \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ 0 & \xi_{\downarrow\downarrow}(\mathbf{k}) & \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \\ \Delta_{\uparrow\uparrow}^*(\mathbf{k}) & \Delta_{\uparrow\downarrow}^*(\mathbf{k}) & -\xi_{\uparrow\uparrow}(\mathbf{k}) & 0 \\ \Delta_{\downarrow\uparrow}^*(\mathbf{k}) & \Delta_{\downarrow\downarrow}^*(\mathbf{k}) & 0 & -\xi_{\downarrow\downarrow}(\mathbf{k}) \end{pmatrix}, \quad (6)$$

which has eigenvalues for the highest quasiparticle band

$$E_1(\mathbf{k}) = \sqrt{\left( \xi_{h-} - \sqrt{\xi_{h+}^2 + |\Delta_S(\mathbf{k})|^2} \right)^2 + |\Delta_T(\mathbf{k})|^2},$$

and for the lowest-energy quasiparticle band,

$$E_2(\mathbf{k}) = \sqrt{\left( \xi_{h-} + \sqrt{\xi_{h+}^2 + |\Delta_S(\mathbf{k})|^2} \right)^2 + |\Delta_T(\mathbf{k})|^2}.$$

The eigenvalues for quasiholes bands are  $E_3(\mathbf{k}) = -E_2(\mathbf{k})$  and  $E_4(\mathbf{k}) = -E_1(\mathbf{k})$ . The term  $\xi_{h-} = [\xi_{\uparrow\uparrow}(\mathbf{k}) - \xi_{\downarrow\downarrow}(\mathbf{k})]/2$  is the average energy difference between the helicity bands  $\xi_{h-} = -|\mathbf{h}_{\text{eff}}(\mathbf{k})|$ ; while the energy  $\xi_{h+} = [\xi_{\uparrow\uparrow}(\mathbf{k}) + \xi_{\downarrow\downarrow}(\mathbf{k})]/2$  is the averaged energy sum of the helicity bands  $\xi_{h+} = K(\mathbf{k}) = \epsilon_{\mathbf{k}} - \mu$ .

Notice that  $E_1(\mathbf{k}) > E_2(\mathbf{k}) \geq 0$ , but that only  $E_2(\mathbf{k})$  can have zeros (nodal regions) corresponding to the locus in momentum space satisfying the following conditions:

- a)  $\xi_{h-} = -\sqrt{\xi_{h+}^2 + |\Delta_S(\mathbf{k})|^2}$ , which corresponds physically to the equality between the effective magnetic field energy  $|\mathbf{h}_{\text{eff}}(\mathbf{k})|$  and the *excitation energy* for the singlet component  $\sqrt{\xi_{h+}^2 + |\Delta_S(\mathbf{k})|^2}$ ; and b)  $|\Delta_T(\mathbf{k})| = 0$ , corresponding to zeros of the triplet component of the order parameter in momentum space.

*Phase Diagram:* Since only  $E_2(\mathbf{k})$  can have zeros, the low energy physics is dominated by this eigenvalue. In the ERD case, where  $|h_{\text{ERD}}(\mathbf{k})| = v|k_x|$ , zeros of  $E_2(\mathbf{k})$  can occur when  $k_x = 0$ , leading to the following cases: (a) two possible lines (rings) of nodes at  $(k_y^2 + k_z^2)/(2m) = \mu + \sqrt{h_z^2 - |\Delta_0|^2}$  for the outer ring, and  $(k_y^2 + k_z^2)/(2m) = \mu - \sqrt{h_z^2 - |\Delta_0|^2}$  for the inner ring, when  $h_z^2 - |\Delta_0|^2 > 0$ ; (b) doubly-degenerate line of nodes at  $(k_y^2 + k_z^2)/(2m) = \mu$  for  $\mu > 0$ , doubly-degenerate point nodes for  $\mu = 0$ , or no-line of nodes for  $\mu < 0$ , when  $h_z^2 - |\Delta_0|^2 = 0$ ; (c) no line of nodes when  $h_z^2 - |\Delta_0|^2 < 0$ . In addition, case (a) can be refined into cases (a2), (a1) and (a0). In case (a2), two rings indeed exist provided that  $\mu > \sqrt{h_z^2 - |\Delta_0|^2}$ . However, the inner ring disappears when  $\mu = \sqrt{h_z^2 - |\Delta_0|^2}$ . In case (a1), there is only one ring when  $|\mu| < \sqrt{h_z^2 - |\Delta_0|^2}$ . In case (a0), the outer ring disappears at  $\mu = -\sqrt{h_z^2 - |\Delta_0|^2}$ , and for  $\mu < -\sqrt{h_z^2 - |\Delta_0|^2}$  no rings exist.

In Fig. 1, we show the phase diagrams of Zeeman field  $h_z/\epsilon_F$  versus interaction parameter  $1/(k_F a_s)$  for spin-orbit coupling  $v/v_F = 0$  (a) and 0.28 (c), as well as induced population imbalance  $P_{\text{ind}}$  versus  $1/(k_F a_s)$  for  $v/v_F = 0$  (b) and 0.28 (d). We label the uniform superfluid phases with zero, one or two rings of nodes as US-0, US-1, and US-2, respectively. Non-uniform (NU) phases also emerge in regions where uniform phases are thermodynamically unstable. Possible NU phases include phase separation, modulated superfluids and supersolid. The US-2/US-1 phase boundary is determined by the condition  $\mu = \sqrt{h_z^2 - |\Delta_0|^2}$ , when  $|h_z| > |\Delta_0|$ ; the

US-0/US-2 boundary is determined by the Clogston-like condition  $|h_z| = |\Delta_0|$  when  $\mu > 0$ , where the gapped US-0 phase disappears leading to the gapless US-2 phase; and the US-0/US-1 phase boundary is determined by  $\mu = -\sqrt{h_z^2 - |\Delta_0|^2}$ , when  $|h_z| > |\Delta_0|$ . Furthermore, within the US-0 boundaries, a crossover line between an indirectly gapped and a directly gapped US-0 phase occurs at  $\mu = 0$ .

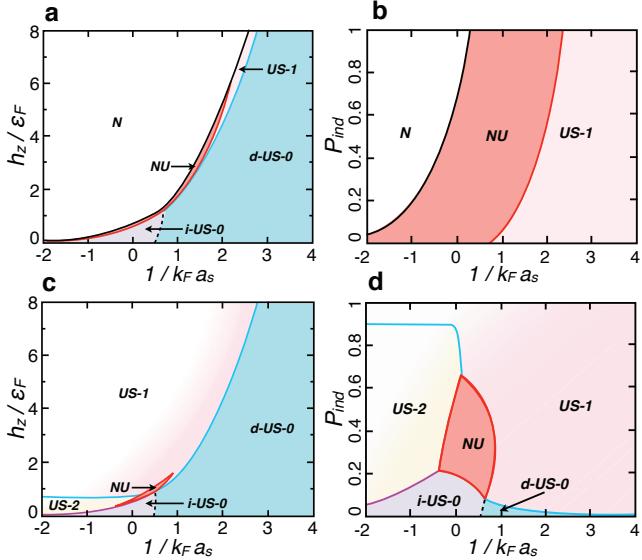


FIG. 1: (color online) Phase diagrams of  $h_z/\epsilon_F$  and  $P_{Ind}$  versus  $1/(k_F a_s)$  for ERD coupling  $v/v_F = 0$  (a), (b) and  $v/v_F = 0.28$  (c), (d). Uniform superfluid phases are labeled as US-0 (gapped, either directly or indirectly), US-1 (gapless with one ring of nodes), and US-2 (gapless with two-rings of nodes). The NU label describes the region where uniform superfluids are unstable.

*Dirac and Majorana fermions:* Changes in nodal structures of the order parameter are associated with bulk topological phase transitions of the Lifshitz class as noted for  $p$ -wave [16, 17] and  $d$ -wave [18, 19] superfluids. Such transitions are possible here because spin-orbit and Zeeman fields induce higher angular momentum pairing in the helicity basis. In the US-1 and US-2 phases near the zeros of  $E_2(\mathbf{k})$ , quasiparticles have linear dispersion and behave as Dirac fermions. The disappearance of nodal regions (rings) correspond to annihilation of Dirac quasiparticles with opposite momenta. The transition from phase US-2 to indirect-gap i-US-0 occurs through the merger of the two-rings at the phase boundary followed by the immediate opening of the indirect gap at finite momentum. However, the transition from phase US-2 to US-1 corresponds to the disappearance of the inner ring through the origin of momenta, and the transition from phase US-1 to the direct-gap d-US-0 corresponds to the disappearance of the last ring also through the origin of momenta.

The last two phase transitions are special because the

zero-momentum quasiparticles at these phase boundaries correspond to Majorana zero energy modes if the phase  $\varphi(\mathbf{k})$  of the spin-orbit field  $h_{ERD}(\mathbf{k}) = |h_{ERD}(\mathbf{k})|e^{i\varphi(\mathbf{k})}$ , where  $\varphi(\mathbf{k}) = \text{sgn}[k_x]\pi/2$  and the phase  $\theta(\mathbf{k})$  of the order parameter  $\Delta_0 = |\Delta_0|e^{i\theta(\mathbf{k})}$  hold the relation at zero momentum:  $\varphi(\mathbf{0}) = -\theta(\mathbf{0}) [\text{mod}(2\pi)]$ . This can be seen from an analysis of the eigenfunctions

$$\Phi_i(\mathbf{k}) = U_{i1,\mathbf{k}}\psi_{\mathbf{k}\uparrow} + U_{i2,\mathbf{k}}\psi_{\mathbf{k}\downarrow} + U_{i3,\mathbf{k}}\psi_{-\mathbf{k}\uparrow}^\dagger + U_{i4,\mathbf{k}}\psi_{-\mathbf{k}\downarrow}^\dagger$$

corresponding to the eigenvalue  $E_i(\mathbf{k})$  (with  $i = 2, 3$ ) where  $U_{ij,\mathbf{k}} = U_{ij}(\mathbf{k})$  are the elements of the unitary matrix  $\mathbf{U}$  that diagonalizes the Hamiltonian  $\tilde{\mathbf{H}}_{sp}$ . The emergence of zero-energy Majorana fermions requires the quasiparticle (quasi-hole) to be its own anti-quasiparticle (anti-quasi-hole):  $\Phi_i^\dagger(\mathbf{k}) = \Phi_i(\mathbf{k})$ . This happens at zero momentum  $\mathbf{k} = \mathbf{0}$ , where the amplitudes  $U_{i1}(\mathbf{0}) = U_{i3}^*(\mathbf{0})$  and  $U_{i2}(\mathbf{0}) = U_{i4}^*(\mathbf{0})$ , leading to the conditions  $\mu^2 = h_z^2 - |\Delta_0|^2$ , and  $\varphi(\mathbf{0}) = -\theta(\mathbf{0}) [\text{mod}(2\pi)]$ , showing that Majorana fermions exist only at the US-1/US-0 and US-2/US-1 phase boundaries. In Fig. 2, we show the Lifshitz transition from US-1 to US-0 phase, where nodal (massless) Dirac Fermions in the US-1 phase become bulk zero-mode Majorana fermions at the US-1/US-0 phase boundary, and then massive Dirac fermions in the US-0 phase.

The commonality between bulk Majorana fermions (found here) and surface Majorana fermions (found in topological insulators or superconductors) is that both exist at boundaries: bulk Majorana zero-energy modes may exist at the phase boundaries between two topologically distinct superfluid phases, while surface Majorana zero-energy modes may exist at the spatial (sample) boundaries of a topologically non-trivial superconductor.

*Lifshitz Transition:* The transition between different superfluid phases occurs without a change in symmetry of the order parameter tensor  $\Delta_{\alpha\beta}(\mathbf{k})$  in the helicity basis, and thus violates the symmetry-based Landau classification of phase transitions. However a finer classification based on topological charges can be made via the construction of topological invariants [16, 20]. The number of rings  $\ell$  corresponds to the topological charge associated with the surfaces of zero-energy quasiparticle excitations. Thus, for the US-0 phase  $\ell = 0$ , while for the US-1 and US-2 phases,  $\ell = 1$  and  $\ell = 2$ , respectively.

*Spectral Function:* An important measurable quantity is the single-particle spectral density [21]  $\mathcal{A}_s(\omega, \mathbf{k}) = -(1/\pi)\text{Im}G_{ss}(i\omega = \omega + i\delta, \mathbf{k})$  for spin  $s = \uparrow, \downarrow$ , which can be extracted from the diagonal elements of the matrix

$$\mathbf{G}(i\omega, \mathbf{k}) = A(i\omega, \mathbf{k})\mathbf{I} + \mathbf{B}(i\omega, \mathbf{k}) \cdot \boldsymbol{\sigma}, \quad (7)$$

where the scalar function is  $A(i\omega, \mathbf{k}) = \{[i\omega + K(\mathbf{k})][(i\omega)^2 - K^2(\mathbf{k}) - |\Delta_0|^2] - |h_{\text{eff}}(\mathbf{k})|^2[i\omega - K(\mathbf{k})]\}/D(i\omega, \mathbf{k})$ , and the vector function components are  $B_{x,y}(i\omega, \mathbf{k}) = \{h_{x,y}(\mathbf{k})[|h_{\text{eff}}(\mathbf{k})|^2 + |\Delta_0|^2 - (i\omega + K(\mathbf{k}))^2]\}/D(i\omega, \mathbf{k})$ , for the transverse and  $B_z(i\omega, \mathbf{k}) = h_z\{[|h_{\text{eff}}(\mathbf{k})|^2 - |\Delta_0|^2 - (i\omega + K(\mathbf{k}))^2]\}/D(i\omega, \mathbf{k})$  for the

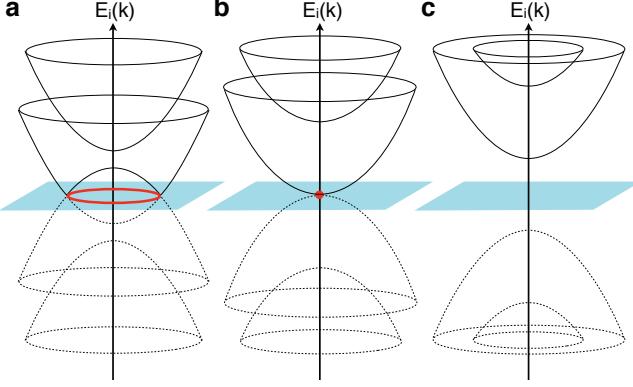


FIG. 2: (color online) Excitation spectra  $E_i(\mathbf{k})$  in the  $(k_x = 0, k_y, k_z)$  plane illustrating the Lifshitz transition: the shrinkage of Dirac rings (US-1 phase) into Majorana zero-energy modes (US-1/US-0 phase boundary) and emergence of massive Dirac fermions (direct-gap d-US-0 phase).

longitudinal parts. Here,  $D(i\omega, \mathbf{k}) = \prod_{j=1}^4 (i\omega - E_j(\mathbf{k}))$  and  $\sigma$  is the vector Pauli matrix. In Fig. 3, we show  $\mathcal{A}_s(\omega, \mathbf{k})$  in the plane of momenta  $k_y - k_z$  with  $k_x = 0$  and  $\omega = 0$  revealing the existence of rings of zero-energy excitations in the US-1 and US-2 phases. In the left panels the spectral densities for the US-1 phase are shown for spin  $\uparrow$  (a), where the ring is brighter than the ring for spin  $\downarrow$  (d). In the middle panels (b) and (e)  $\mathcal{A}_s(\omega, \mathbf{k})$  at the US-1/US-0 phase boundary is shown revealing the Majorana zero-energy mode. In the right panels (c) and (f) the spectral densities for the US-0 phase vanish at  $\omega = 0$ , since this phase is fully gapped.

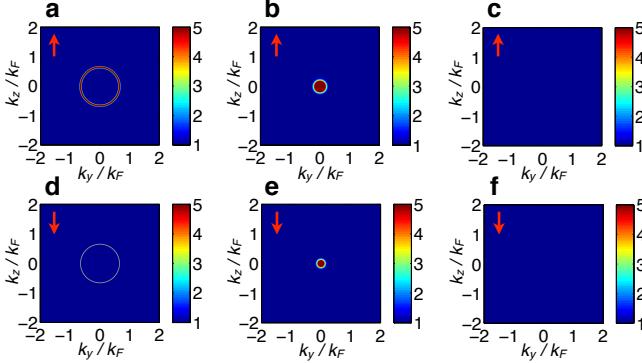


FIG. 3: (color online) The zero-energy spectral density  $\mathcal{A}_s(\omega = 0, k_x = 0, k_y, k_z)$  at  $1/(k_F a_s) = 1.0$  and  $v/v_F = 0.28$  is shown in (a) and (d) for the US-1 phase with  $h_z/\epsilon_F = 1.75$ , in (b) and (e) for the US-1/US-0 phase boundary with  $h_z/\epsilon_F = 1.59$ , and in (c) and (f) for the direct-gap d-US-0 phase with  $h_z/\epsilon_F = 1.44$ .

*Momentum Distribution:* A spectroscopic quantity that is routinely measured is the momentum distribution

$$n_s(\mathbf{k}) = T \sum_{i\omega} [A(i\omega, \mathbf{k}) \pm B_z(i\omega, \mathbf{k})], \quad (8)$$

where the  $+$ ( $-$ ) sign corresponds to spin  $\uparrow$  ( $\downarrow$ ). Since  $n_s(\mathbf{k})$  depends only on the energy spectrum  $E_j(\mathbf{k})$  and its derivatives, it is an even function of momentum  $\mathbf{k}$ . In Fig. 4, we show  $n_s(\mathbf{k})$  for  $1/(k_F a_s) = 1.0$  and  $v/v_F = 0.28$  at the US-1 phase with  $h_z/\epsilon_F = 1.75$  (top panels), at the US-1/US-0 phase boundary with  $h_z/\epsilon_F = 1.59$  (middle panels), at the US-0 phase with  $h_z/\epsilon_F = 1.44$  (lower panels). The left-most (right-most) panels show the momentum distribution for spin  $\uparrow$  ( $\downarrow$ ) at  $k_z = 0$  versus  $k_y$  (solid-blue line) and versus  $k_x$  (dotted-red line). It is very important to note the discontinuity of  $n_s(\mathbf{k})$  at the location of the ring of zero-energy excitations in the US-1 phase (top panels), its change in behavior as bulk Majorana fermions emerge at  $\mathbf{k} = 0$  (middle panels), and the transition to a direct-gap d-US-0 phase (bottom panels), where the ring of nodes has disappeared leading to smooth momentum distributions  $n_s(\mathbf{k})$ .

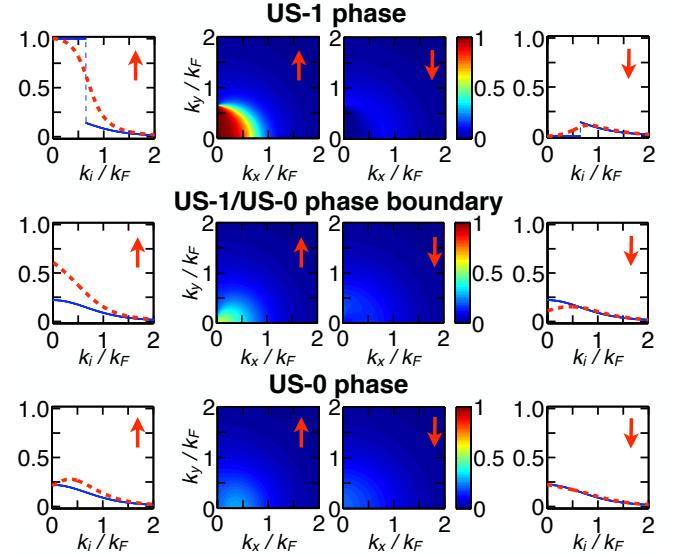


FIG. 4: (color online) Momentum distribution  $n_s(\mathbf{k})$  for  $1/(k_F a_s) = 1.0$  and  $v/v_F = 0.28$  at the US-1 phase with  $h_z/\epsilon_F = 1.75$  (top panels), at the US-1/US-0 phase boundary with  $h_z/\epsilon_F = 1.59$  (middle panels), at the US-0 phase with  $h_z/\epsilon_F = 1.44$  (lower panels). The left-most (right-most) panels show the momentum distribution for spin  $\uparrow$  ( $\downarrow$ ) at  $k_z = 0$  versus  $k_y$  (solid-blue line) and versus  $k_x$  (dotted-red line).

*Summary:* We showed that the presence of simultaneous Zeeman and spin-orbit fields induces higher angular momentum pairing in the helicity basis, and identified topological phase transitions of the Lifshitz class via the existence of: rings of nodes in the excitation spectra, Dirac quasiparticles, bulk Majorana zero-energy modes, and topological charges. Lastly, we characterized different topological phases via experimentally measurable quantities such as the spectral function and momentum distribution, and concluded that Lifshitz is the lord of the rings.

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